Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

PHILOSOPHICAL TRANSACTIONS A

rsta.royalsocietypublishing.org

Research

Check for updates

Cite this article: Griffiths DJ. 2018 A catalogue of hidden momenta. *Phil. Trans. R. Soc. A* **376**: 20180043.
http://dx.doi.org/10.1098/rsta.2018.0043

Accepted: 14 May 2018

One contribution of 13 to a theme issue 'Celebrating 125 years of Oliver Heaviside's 'Electromagnetic Theory''.

Subject Areas:
electromagnetism

Keywords:
classical electrodynamics, electromagnetic momentum, hidden momentum, electric dipoles, magnetic dipoles, magnetic monopoles

Author for correspondence:
David J. Griffiths
e-mail: griffith@reed.edu

A catalogue of hidden momenta

David J. Griffiths

Department of Physics, Reed College, Portland, OR 97202, USA

DJG, 0000-0002-4406-4781

Electromagnetic fields carry momentum: $\mathbf{P}_{\mathrm{em}} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, \mathrm{d}\tau$. But if the centre of energy of a (localized) system is at rest, its *total* momentum must be zero. The compensating term has come to be called 'hidden' momentum: $\mathbf{P}_{\mathrm{h}} = -\mathbf{P}_{\mathrm{em}}$. It is (typically) ordinary *mechanical* momentum, relativistic in nature, and is 'hidden' only in the sense that it is not associated with motion of the system as a whole—only with that of its constituent parts. This article develops a catalogue of field momenta and hidden momenta for ideal electric and magnetic dipoles—both the 'standard' variety made from electric charges and currents and the 'anomalous' variety made from hypothetical magnetic monopoles and their currents—in the presence of electric and magnetic fields (which themselves may be produced by 'standard' or 'anomalous' sources).

This article is part of the theme issue 'Celebrating 125 years of Oliver Heaviside's 'Electromagnetic Theory''.

1. Electric and magnetic dipoles

In the static case, Maxwell's equations read

$$\left.\begin{array}{ll} \text{(a) } \boldsymbol{\nabla}\cdot\mathbf{E} = \left(\dfrac{1}{\epsilon_0}\right)\rho, & \text{(c) } \boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}, \\ \text{(b) } \boldsymbol{\nabla}\cdot\mathbf{B} = 0, & \text{(d) } \boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}, \end{array}\right\} \tag{1.1}$$

where ρ is the electric charge density and $\mathbf{J}$ is the electric current density. In a world with magnetic monopoles, there would also exist electromagnetic fields sourced by magnetic charges ($\tilde{\rho}$) and their currents ($\tilde{\mathbf{J}}$):

$$\left.\begin{array}{ll} \text{(a) } \boldsymbol{\nabla}\cdot\tilde{\mathbf{E}} = 0, & \text{(c) } \boldsymbol{\nabla}\times\tilde{\mathbf{E}} = -\mu_0\tilde{\mathbf{J}}, \\ \text{(b) } \boldsymbol{\nabla}\cdot\tilde{\mathbf{B}} = \mu_0\tilde{\rho}, & \text{(d) } \boldsymbol{\nabla}\times\tilde{\mathbf{B}} = \mathbf{0}. \end{array}\right\} \tag{1.2}$$

(I will use a tilde for these 'anomalous' sources and fields, to distinguish them from the 'standard'

THE ROYAL SOCIETY PUBLISHING

© 2018 The Author(s) Published by the Royal Society. All rights reserved.

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

variety associated with ordinary electric charge.) It follows (by applying the divergence to equations (1.1(*d*)) and (1.2(*c*)) that the electric and magnetic currents are divergenceless:

$$\boldsymbol{\nabla}\cdot\mathbf{J}=0,\quad \boldsymbol{\nabla}\cdot\tilde{\mathbf{J}}=0, \tag{1.3}$$

(the associated charges are locally conserved).

The force on an electric charge (q) moving with velocity $\mathbf{v}$ is given by the Lorentz force law:

$$\mathbf{F}=q[\mathbf{E}'+(\mathbf{v}\times\mathbf{B}')], \tag{1.4}$$

where $\mathbf{E}'=\mathbf{E}+\tilde{\mathbf{E}}$ is the *total* electric field (standard plus anomalous), and $\mathbf{B}'=\mathbf{B}+\tilde{\mathbf{B}}$. Likewise, the force on a magnetic monopole ($\tilde{q}$) is

$$\mathbf{F}=\tilde{q}[\mathbf{B}'-\epsilon_0\mu_0(\mathbf{v}\times\mathbf{E}')]. \tag{1.5}$$

(The slight asymmetry in all these formulae is an unfortunate artefact of the SI system, and would not appear in Gaussian units.)

The fields can be expressed in terms of scalar and vector potentials:[1]

$$\left.\begin{aligned}&\mathbf{E}=-\boldsymbol{\nabla}V,\qquad \mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A},\\ \text{and}\qquad &\tilde{\mathbf{E}}=-\boldsymbol{\nabla}\times\tilde{\mathbf{A}},\quad \tilde{\mathbf{B}}=-\boldsymbol{\nabla}\tilde{V}.\end{aligned}\right\} \tag{1.6}$$

Adopting the gauge condition

$$\boldsymbol{\nabla}\cdot\mathbf{A}=0,\quad \boldsymbol{\nabla}\cdot\tilde{\mathbf{A}}=0, \tag{1.7}$$

Maxwell's equations become

$$\left.\begin{aligned}&\nabla^2V=-\frac{1}{\epsilon_0}\rho,\quad \nabla^2\mathbf{A}=-\mu_0\mathbf{J},\\ \text{and}\qquad &\nabla^2\tilde{V}=-\mu_0\tilde{\rho},\quad \nabla^2\tilde{\mathbf{A}}=-\mu_0\tilde{\mathbf{J}}.\end{aligned}\right\} \tag{1.8}$$

For localized charge and current configurations, it follows that

$$V(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\tau',\quad \mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\tau', \tag{1.9}$$

and

$$\tilde{V}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\tilde{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\tau',\quad \tilde{\mathbf{A}}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\tilde{\mathbf{J}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\tau'. \tag{1.10}$$

The electric dipole moment of an electric charge distribution (ρ) is defined by

$$\mathbf{p}\equiv\int\mathbf{r}\rho\,\mathrm{d}\tau, \tag{1.11}$$

where $\mathbf{r}$ is the vector from the origin to the volume element $\mathrm{d}\tau$. The magnetic dipole moment of an electric current configuration ($\mathbf{J}$) is

$$\mathbf{m}\equiv\frac{1}{2}\int(\mathbf{r}\times\mathbf{J})\,\mathrm{d}\tau. \tag{1.12}$$

I shall call these 'standard' dipoles, to distinguish them from the hypothetical 'anomalous' variety associated with monopole charges and currents:

$$\tilde{\mathbf{m}}\equiv\int\mathbf{r}\tilde{\rho}\,\mathrm{d}\tau, \tag{1.13}$$

and

$$\tilde{\mathbf{p}}\equiv-\frac{\epsilon_0\mu_0}{2}\int(\mathbf{r}\times\tilde{\mathbf{J}})\,\mathrm{d}\tau. \tag{1.14}$$

[1]Many of the equations in §1 are taken from [1] and are recapitulated here in order to make the paper self-contained. However, I have reversed an unfortunate sign convention in the definition of $\tilde{\mathbf{A}}$.

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

If (as we shall always assume) the dipoles are *neutral*,

$$\int \rho \, d\tau = 0, \quad \int \tilde{\rho} \, d\tau = 0, \tag{1.15}$$

then $\mathbf{p}$ and $\tilde{\mathbf{m}}$ are independent of the choice of origin. It follows from equation (1.3) that

$$\begin{aligned} 0 = \int r_i(\boldsymbol{\nabla} \cdot \mathbf{J}) \, d\tau = \int r_i(\nabla_j J_j) \, d\tau &= \int \nabla_j(r_i J_j) \, d\tau - \int (\nabla_j r_i) J_j \, d\tau \\ &= -\int \delta_{ji} J_j \, d\tau = -\int J_j \, d\tau \end{aligned} \tag{1.16}$$

(repeated indices are to be summed from 1 to 3; we assume that charge and current distributions are *localized*, so all boundary terms coming from integration by parts vanish). Thus,

$$\int \mathbf{J} \, d\tau = \mathbf{0}, \tag{1.17}$$

and in this case, the standard magnetic dipole moment ($\mathbf{m}$) is independent of origin. By the same reasoning,

$$\int \tilde{\mathbf{J}} \, d\tau = \mathbf{0}, \tag{1.18}$$

and $\tilde{\mathbf{p}}$ is independent of origin.

Similarly,

$$\begin{aligned} 0 = \int r_i r_j(\nabla_k J_k) \, d\tau = -\int [\nabla_k(r_i r_j)] J_k \, d\tau &= -\int (r_i \delta_{jk} + r_j \delta_{ik}) J_k \, d\tau \\ &= -\int (r_i J_j + r_j J_i) \, d\tau. \end{aligned} \tag{1.19}$$

On the other hand, from equation (1.12),

$$\begin{aligned} \epsilon_{ijk} m_k = \frac{1}{2}(\epsilon_{ijk}\epsilon_{klm}) \int r_l J_m \, d\tau &= \frac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) \int r_l J_m \, d\tau \\ &= \frac{1}{2}\int (r_i J_j - r_j J_i) \, d\tau = \int r_i J_j \, d\tau. \end{aligned} \tag{1.20}$$

By the same token,

$$\epsilon_{ijk}\tilde{p}_k = -\mu_0\epsilon_0 \int r_i \tilde{J}_j \, d\tau. \tag{1.21}$$

In general, the charge and current configurations constituting a physical dipole will be distributed over some finite region of space. However, we shall from now on confine our attention to 'ideal' dipoles, localized at a single point. More precisely, we are interested in the *limiting case*, in which the size of the dipole shrinks to zero.

To compute the potential of such a dipole, we note that

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + r'^2 - 2\mathbf{r} \cdot \mathbf{r}'}. \tag{1.22}$$

For an *ideal* dipole at the origin, the charge (or current) distribution at $\mathbf{r}'$ vanishes except at $\mathbf{r}' \to 0$, so (in equations (1.9) and (1.10)) we may safely confine our attention to the region $r' \ll r$, for which the binomial expansion gives

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{1}{r}\left(1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2}\right). \tag{1.23}$$

In the case of a standard electric dipole,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{1}{r}\int \rho(\mathbf{r}') \, d\tau' + \frac{\mathbf{r}}{r^3} \cdot \int \mathbf{r}'\rho(\mathbf{r}') \, d\tau'\right\}, \tag{1.24}$$

or (using equations (1.11) and (1.15)),

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}. \tag{1.25}$$

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

Similarly, the potential of a non-standard magnetic dipole is

$$\tilde{V}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\tilde{\mathbf{m}} \cdot \mathbf{r}}{r^3}. \tag{1.26}$$

For a standard magnetic dipole,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\frac{1}{r}\int \mathbf{J}(\mathbf{r}')\,\mathrm{d}\tau' + \frac{1}{r^3}\int (\mathbf{r}\cdot\mathbf{r}')\mathbf{J}(\mathbf{r}')\,\mathrm{d}\tau'\right\}, \tag{1.27}$$

or (using equations (1.17) and (1.20))

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}. \tag{1.28}$$

Likewise, for an anomalous electric dipole:

$$\tilde{\mathbf{A}}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\frac{\tilde{\mathbf{p}}\times\mathbf{r}}{r^3}. \tag{1.29}$$

To determine the field of a dipole, we take the gradient or curl of the pertinent potential (equation (1.6)). This requires some care, however, because the dipole potentials are very singular at the origin. In general [2, eqn 6],

$$\nabla_i\left(\frac{r_j}{r^3}\right) = \frac{1}{r^3}\left(\delta_{ij} - 3\frac{r_i r_j}{r^2}\right) + \frac{4\pi}{3}\delta_{ij}\delta^3(\mathbf{r}), \tag{1.30}$$

so for any constant vector **a**,

$$\nabla\left(\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right) = \frac{1}{r^3}\left(\mathbf{a} - 3\frac{\mathbf{r}(\mathbf{a}\cdot\mathbf{r})}{r^2}\right) + \frac{4\pi}{3}\mathbf{a}\delta^3(\mathbf{r}) \tag{1.31}$$

and

$$\nabla\times\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = -\frac{1}{r^3}\left(\mathbf{a} - 3\frac{\mathbf{r}(\mathbf{a}\cdot\mathbf{r})}{r^2}\right) + \frac{8\pi}{3}\mathbf{a}\delta^3(\mathbf{r}). \tag{1.32}$$

Using these two identities, we find

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(3\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{p})}{r^2} - \mathbf{p}\right) - \frac{\mathbf{p}}{3\epsilon_0}\delta^3(\mathbf{r}), \tag{1.33}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left(3\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{m})}{r^2} - \mathbf{m}\right) + \frac{2\mu_0\mathbf{m}}{3}\delta^3(\mathbf{r}), \tag{1.34}$$

$$\tilde{\mathbf{E}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(3\frac{\mathbf{r}(\mathbf{r}\cdot\tilde{\mathbf{p}})}{r^2} - \tilde{\mathbf{p}}\right) + \frac{2\tilde{\mathbf{p}}}{3\epsilon_0}\delta^3(\mathbf{r}) \tag{1.35}$$

and

$$\tilde{\mathbf{B}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left(3\frac{\mathbf{r}(\mathbf{r}\cdot\tilde{\mathbf{m}})}{r^2} - \tilde{\mathbf{m}}\right) - \frac{\mu_0\tilde{\mathbf{m}}}{3}\delta^3(\mathbf{r}). \tag{1.36}$$

The delta-function terms are often left out, because one is usually interested in the field at some remove from the dipole; what remains has the same form in all four cases. This 'universal' part holds outside a sphere of vanishingly small radius; the delta-function describes the field inside this sphere. Although the latter contributes only at one point, it is essential for the internal consistency of the theory.

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

It is useful to note that this entire theory is invariant under the following duality transformation:

$\rho \to \frac{1}{c}\tilde{\rho}$	$\mathbf{J} \to \frac{1}{c}\tilde{\mathbf{J}}$	$\tilde{\rho} \to -c\rho$	$\tilde{\mathbf{J}} \to -c\mathbf{J}$
$\mathbf{E} \to c\tilde{\mathbf{B}}$	$\mathbf{B} \to -\frac{1}{c}\tilde{\mathbf{E}}$	$\tilde{\mathbf{E}} \to c\mathbf{B}$	$\tilde{\mathbf{B}} \to -\frac{1}{c}\mathbf{E}$
$V \to c\tilde{V}$	$\mathbf{A} \to \frac{1}{c}\tilde{\mathbf{A}}$	$\tilde{V} \to -\frac{1}{c}V$	$\tilde{\mathbf{A}} \to -c\mathbf{A}$
$\mathbf{p} \to \frac{1}{c}\tilde{\mathbf{m}}$	$\mathbf{m} \to -c\tilde{\mathbf{p}}$	$\tilde{\mathbf{p}} \to \frac{1}{c}\mathbf{m}$	$\tilde{\mathbf{m}} \to -c\mathbf{p}$

2. Field momentum

The (linear) momentum in electromagnetic fields is

$$\mathbf{P}_{\mathrm{em}} = \epsilon_0 \int (\mathbf{E}' \times \mathbf{B}')\,\mathrm{d}\tau \tag{2.1}$$

(the fields could be standard or anomalous, or—in principle—some of each). It is often more convenient to express this equation in terms of potentials; the resulting formula depends on the nature of the sources:

1. *Standard electric and magnetic fields*: $\mathbf{E} = -\boldsymbol{\nabla} V$, so

$$\begin{aligned}\mathbf{P}_{\mathrm{em}} &= -\epsilon_0 \int [(\boldsymbol{\nabla} V) \times \mathbf{B}]\,\mathrm{d}\tau = -\epsilon_0 \left[\int \boldsymbol{\nabla} \times (V\mathbf{B})\,\mathrm{d}\tau - \int V(\boldsymbol{\nabla} \times \mathbf{B})\,\mathrm{d}\tau\right] \\ &= \mu_0\epsilon_0 \int V(\mathbf{r})\mathbf{J}(\mathbf{r})\,\mathrm{d}\tau. \end{aligned} \tag{2.2}$$

   On the other hand, since[2] (equation (1.9))

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\tau' \quad \text{and} \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\tau', \\ \mathbf{P}_{\mathrm{em}} &= \mu_0\epsilon_0 \frac{1}{4\pi\epsilon_0}\iint \frac{\rho(\mathbf{r}')\mathbf{J}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\tau'\,\mathrm{d}\tau = \int \left\{\frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r})}{|\mathbf{r}'-\mathbf{r}|}\,\mathrm{d}\tau\right\}\rho(\mathbf{r}')\,\mathrm{d}\tau' \\ &= \int \rho(\mathbf{r})\mathbf{A}(\mathbf{r})\,\mathrm{d}\tau. \end{aligned} \tag{2.3}$$

2. *Standard electric field and anomalous magnetic field*: The first line of equation (2.2) still holds, but since $\boldsymbol{\nabla} \times \tilde{\mathbf{B}} = \mathbf{0}$, the field momentum is zero:

$$\mathbf{P}_{\mathrm{em}} = \mathbf{0}. \tag{2.4}$$

[2]Of course, you could go back to equation (2.1) and insert $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, but it's a little easier to work from equation (2.2).

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

3. *Anomalous electric and magnetic fields*:

$$\begin{aligned}\mathbf{P}_{\rm em} &= \epsilon_0 \int (\tilde{\mathbf{E}} \times \tilde{\mathbf{B}})\,\mathrm{d}\tau = -\epsilon_0 \int (\tilde{\mathbf{E}} \times \boldsymbol{\nabla}\tilde{V})\,\mathrm{d}\tau = -\epsilon_0 \int \tilde{V}(\boldsymbol{\nabla} \times \tilde{\mathbf{E}})\,\mathrm{d}\tau \\ &= \mu_0\epsilon_0 \int \tilde{V}(\mathbf{r})\tilde{\mathbf{J}}(\mathbf{r})\,\mathrm{d}\tau, \end{aligned} \tag{2.5}$$

and since (equation (1.10))

$$\left.\begin{aligned} &\tilde{V}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\tilde{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\tau' \quad \text{and} \quad \tilde{\mathbf{A}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\tilde{\mathbf{J}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\tau', \\ \text{and hence} \qquad &\mathbf{P}_{\rm em} = \mu_0\epsilon_0 \int \tilde{\rho}(\mathbf{r})\tilde{\mathbf{A}}(\mathbf{r})\,\mathrm{d}\tau. \end{aligned}\right\} \tag{2.6}$$

4. *Anomalous electric field and standard magnetic field*:

$$\begin{aligned}\mathbf{P}_{\rm em} &= \epsilon_0 \int \tilde{\mathbf{E}} \times (\boldsymbol{\nabla} \times \mathbf{A})\,\mathrm{d}\tau \\ &= -\epsilon_0 \int \Big[\mathbf{A} \times (\boldsymbol{\nabla} \times \tilde{\mathbf{E}}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\tilde{\mathbf{E}} + (\tilde{\mathbf{E}}\cdot\boldsymbol{\nabla})\mathbf{A}\Big]\,\mathrm{d}\tau \\ &= \mu_0\epsilon_0 \int (\mathbf{A} \times \tilde{\mathbf{J}})\,\mathrm{d}\tau. \end{aligned} \tag{2.7}$$

I used the fact that the ith component of $\int (\mathbf{A}\cdot\boldsymbol{\nabla})\tilde{\mathbf{E}}\,\mathrm{d}\tau$ is

$$\int \mathbf{A}\cdot\boldsymbol{\nabla}(\tilde{E}_i)\,\mathrm{d}\tau = -\int (\boldsymbol{\nabla}\cdot\mathbf{A})\tilde{E}_i\,\mathrm{d}\tau = 0, \tag{2.8}$$

and the same goes for $\int (\tilde{\mathbf{E}}\cdot\boldsymbol{\nabla})\mathbf{A}\,\mathrm{d}\tau$. Finally, using the now-familiar trick,

$$\begin{aligned}\mathbf{P}_{\rm em} &= \mu_0\epsilon_0 \int (\mathbf{A} \times \tilde{\mathbf{J}})\,\mathrm{d}\tau = \mu_0\epsilon_0 \iint \frac{\mu_0}{4\pi}\frac{\mathbf{J}(\mathbf{r}') \times \tilde{\mathbf{J}}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\tau'\,\mathrm{d}\tau \\ &= \mu_0\epsilon_0 \int (\mathbf{J} \times \tilde{\mathbf{A}})\,\mathrm{d}\tau. \end{aligned} \tag{2.9}$$

I summarize these results as follows:

$\mathbf{E}$ and $\mathbf{B}$	$\mathbf{P}_{\rm em} = \mu_0\epsilon_0 \int V\,\mathbf{J}\,d\tau = \int \rho\,\mathbf{A}\,d\tau$
$\mathbf{E}$ and $\tilde{\mathbf{B}}$	$\mathbf{P}_{\rm em} = \mathbf{0}$
$\tilde{\mathbf{E}}$ and $\mathbf{B}$	$\mathbf{P}_{\rm em} = \mu_0\epsilon_0 \int (\mathbf{A} \times \tilde{\mathbf{J}})\,d\tau = -\mu_0\epsilon_0 \int (\tilde{\mathbf{A}} \times \mathbf{J})\,d\tau$
$\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$	$\mathbf{P}_{\rm em} = \mu_0\epsilon_0 \int \tilde{V}\,\tilde{\mathbf{J}}\,d\tau = \mu_0\epsilon_0 \int \tilde{\rho}\,\tilde{\mathbf{A}}\,d\tau$

Now we will use these formulae to determine the field momenta of electric and magnetic dipoles (both standard and anomalous) in external electric and magnetic fields (both standard

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

and anomalous). The dipoles are at rest (we might as well put them at the origin), and since they occupy an infinitesimal volume we can expand the external potentials:

$$V(\mathbf{r}) = V(\mathbf{0}) + \mathbf{r}\cdot(\boldsymbol{\nabla}_0 V) = V(\mathbf{0}) - \mathbf{r}\cdot\mathbf{E}(\mathbf{0}); \quad \mathbf{A}(\mathbf{r}) = \mathbf{A}(\mathbf{0}) + (\mathbf{r}\cdot\boldsymbol{\nabla}_0)\mathbf{A}; \tag{2.10}$$

and

$$\tilde{V}(\mathbf{r}) = \tilde{V}(\mathbf{0}) + \mathbf{r}\cdot(\boldsymbol{\nabla}_0 \tilde{V}) = \tilde{V}(\mathbf{0}) - \mathbf{r}\cdot\tilde{\mathbf{B}}(\mathbf{0}); \quad \tilde{\mathbf{A}}(\mathbf{r}) = \tilde{\mathbf{A}}(\mathbf{0}) + (\mathbf{r}\cdot\boldsymbol{\nabla}_0)\tilde{\mathbf{A}}. \tag{2.11}$$

(Here $\boldsymbol{\nabla}_0$ means 'evaluate the derivatives at $\mathbf{r} = \mathbf{0}$,' and we do not need any higher-order terms.)

1. *Standard electric dipole in standard magnetic field.* Use equation (2.3):

$$\begin{aligned}\mathbf{P}_{\text{em}} &= \int \rho(\mathbf{r})[\mathbf{A}(\mathbf{0}) + (\mathbf{r}\cdot\boldsymbol{\nabla}_0)\mathbf{A}]\,\mathrm{d}\tau \\ &= \left\{\int \rho(\mathbf{r})\,\mathrm{d}\tau\right\}\mathbf{A}(\mathbf{0}) + \left(\left\{\int \mathbf{r}\rho(\mathbf{r})\,\mathrm{d}\tau\right\}\cdot\boldsymbol{\nabla}_0\right)\mathbf{A} \\ &= (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{A}.\end{aligned} \tag{2.12}$$

   (I dropped the subscript on $\boldsymbol{\nabla}$; it is to be evaluated at the location of the dipole.)

2. *Standard magnetic dipole in standard electric field.* Use equation (2.2):

$$\mathbf{P}_{\text{em}} = \mu_0\epsilon_0\left\{V(0)\int \mathbf{J}(\mathbf{r})\,\mathrm{d}\tau - E_j(\mathbf{0})\int \mathbf{r}_j\mathbf{J}\,\mathrm{d}\tau\right\}. \tag{2.13}$$

   But $\int \mathbf{J}\,\mathrm{d}\tau = \mathbf{0}$ and $\int r_i J_j\,\mathrm{d}\tau = \epsilon_{ijk}m_k$, so the jth component is

$$P_{\text{em}j} = -\mu_0\epsilon_0 E_i(\mathbf{0})\int \mathbf{r}_i J_j\,\mathrm{d}\tau = -\mu_0\epsilon_0 E_i(\mathbf{0})\epsilon_{ijk}m_k. \tag{2.14}$$

   (I used equation (1.20)) and therefore

$$\mathbf{P}_{\text{em}} = \mu_0\epsilon_0(\mathbf{E}\times\mathbf{m}). \tag{2.15}$$

3. *Standard electric dipole in anomalous magnetic field, or anomalous magnetic dipole in standard electric field.* Equation (2.4) says

$$\mathbf{P}_{\text{em}} = \mathbf{0}. \tag{2.16}$$

4. *Anomalous electric dipole in anomalous magnetic field.* Use equation (2.5):

$$\left.\begin{aligned}\mathbf{P}_{\text{em}} &= \mu_0\epsilon_0\int[\tilde{V}(\mathbf{0}) - \mathbf{r}\cdot\tilde{\mathbf{B}}(\mathbf{0})]\tilde{\mathbf{J}}\,\mathrm{d}\tau. \\ P_{\text{em}i} &= -\mu_0\epsilon_0\tilde{B}_j(\mathbf{0})\int r_j\tilde{J}_i\,\mathrm{d}\tau = -\mu_0\epsilon_0\tilde{B}_j(\mathbf{0})\left(-\frac{1}{\mu_0\epsilon_0}\right)\epsilon_{jik}\tilde{p}_k \\ &= -\epsilon_{ijk}\tilde{B}_j(\mathbf{0})\tilde{p}_k, \\ \text{so}\qquad \mathbf{P}_{\text{em}} &= -\tilde{\mathbf{B}}\times\tilde{\mathbf{p}}.\end{aligned}\right\} \tag{2.17}$$

5. *Anomalous magnetic dipole in anomalous electric field.* Use equation (2.6):

$$\left.\begin{aligned}\mathbf{P}_{\text{em}} &= \mu_0\epsilon_0\int[\tilde{\mathbf{A}}(\mathbf{0}) + (\mathbf{r}\cdot\boldsymbol{\nabla}_0)\tilde{\mathbf{A}}]\tilde{\rho}(\mathbf{r})\,\mathrm{d}\tau. \\ P_{\text{em}i} &= \mu_0\epsilon_0(\nabla_{0j}\tilde{A}_i)\int r_j\tilde{\rho}(\mathbf{r})\,\mathrm{d}\tau = \mu_0\epsilon_0(\nabla_{0j}\tilde{A}_i)\tilde{m}_j \\ \text{so}\qquad \mathbf{P}_{\text{em}} &= \mu_0\epsilon_0(\tilde{\mathbf{m}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{A}}.\end{aligned}\right\} \tag{2.18}$$

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

6. *Anomalous electric dipole in standard magnetic field.* Use equation (2.7):

$$\left.\begin{aligned} \mathbf{P}_{\mathrm{em}} &= \mu_0\epsilon_0\int[\mathbf{A}(\mathbf{0}) + (\mathbf{r}\cdot\boldsymbol{\nabla}_0)\mathbf{A}]\times\tilde{\mathbf{J}}(\mathbf{r})\,\mathrm{d}\tau. \\ P_{\mathrm{em}i} &= \mu_0\epsilon_0\epsilon_{ijk}(\nabla_{0l}A_j)\int r_l\tilde{J}_k\,\mathrm{d}\tau = \mu_0\epsilon_0\epsilon_{ijk}(\nabla_{0l}A_j)\left(-\frac{1}{\mu_0\epsilon_0}\right)\epsilon_{lkm}\tilde{p}_m \\ &= -(\delta_{im}\delta_{jl}-\delta_{il}\delta_{jm})(\nabla_{0l}A_j)\tilde{p}_m = -(\nabla_0\cdot\mathbf{A})\tilde{p}_i + (\nabla_{0i}A_j)\tilde{p}_j, \\ \text{so}\qquad \mathbf{P}_{\mathrm{em}} &= \tilde{p}_j\boldsymbol{\nabla}A_j = \boldsymbol{\nabla}(\tilde{\mathbf{p}}\cdot\mathbf{A}) = \tilde{\mathbf{p}}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\tilde{\mathbf{p}}\cdot\boldsymbol{\nabla})\mathbf{A} \\ &= (\tilde{\mathbf{p}}\times\mathbf{B}) + (\tilde{\mathbf{p}}\cdot\boldsymbol{\nabla})\mathbf{A}. \end{aligned}\right\} \tag{2.19}$$

(The form $\boldsymbol{\nabla}(\tilde{\mathbf{p}}\cdot\mathbf{A})$ is tidy but dangerous: the derivative does *not* act on $\tilde{p}$, only on $\mathbf{A}$.)

7. *Standard magnetic dipole in anomalous electric field.* Use equation (2.9):

$$\left.\begin{aligned} \mathbf{P}_{\mathrm{em}} &= -\mu_0\epsilon_0\int[\tilde{\mathbf{A}}(\mathbf{0}) + (\mathbf{r}\cdot\boldsymbol{\nabla}_0)\tilde{\mathbf{A}}]\times\mathbf{J}(\mathbf{r})\,\mathrm{d}\tau. \\ P_{\mathrm{em}i} &= -\mu_0\epsilon_0\epsilon_{ijk}(\nabla_{0l}\tilde{A}_j)\int r_lJ_k\,\mathrm{d}\tau = -\mu_0\epsilon_0\epsilon_{ijk}(\nabla_{0l}\tilde{A}_j)\epsilon_{lkm}m_m \\ &= -\mu_0\epsilon_0(\delta_{im}\delta_{jl}-\delta_{il}\delta_{jm})(\nabla_{0l}\tilde{A}_j)m_m = \mu_0\epsilon_0(\nabla_{0i}\tilde{A}_j)m_j \\ \text{so}\qquad \mathbf{P}_{\mathrm{em}} &= \mu_0\epsilon_0 m_j\boldsymbol{\nabla}\tilde{A}_j = \mu_0\epsilon_0\boldsymbol{\nabla}(\mathbf{m}\cdot\tilde{\mathbf{A}}) = \mu_0\epsilon_0[\mathbf{m}\times(\boldsymbol{\nabla}\times\tilde{\mathbf{A}}) + (\mathbf{m}\cdot\boldsymbol{\nabla})\tilde{\mathbf{A}}] \\ &= \mu_0\epsilon_0[-\mathbf{m}\times\tilde{\mathbf{E}} + (\mathbf{m}\cdot\boldsymbol{\nabla})\tilde{\mathbf{A}}]. \end{aligned}\right\} \tag{2.20}$$

I summarize the results as follows:

$\mathbf{p}$ in $\mathbf{B}$	$\mathbf{P}_{\mathrm{em}} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{A}$	$\tilde{\mathbf{p}}$ in $\mathbf{B}$	$\mathbf{P}_{\mathrm{em}} = (\tilde{\mathbf{p}}\times\mathbf{B}) + (\tilde{\mathbf{p}}\cdot\boldsymbol{\nabla})\mathbf{A}$
$\mathbf{m}$ in $\mathbf{E}$	$\mathbf{P}_{\mathrm{em}} = -\mu_0\epsilon_0(\mathbf{m}\times\mathbf{E})$	$\tilde{\mathbf{m}}$ in $\mathbf{E}$	$\mathbf{P}_{\mathrm{em}} = \mathbf{0}$
$\mathbf{p}$ in $\tilde{\mathbf{B}}$	$\mathbf{P}_{\mathrm{em}} = \mathbf{0}$	$\tilde{\mathbf{p}}$ in $\tilde{\mathbf{B}}$	$\mathbf{P}_{\mathrm{em}} = \tilde{\mathbf{p}}\times\tilde{\mathbf{B}}$
$\mathbf{m}$ in $\tilde{\mathbf{E}}$	$\mathbf{P}_{\mathrm{em}} = -\frac{1}{c^2}[\mathbf{m}\times\tilde{\mathbf{E}} - (\mathbf{m}\cdot\boldsymbol{\nabla})\tilde{\mathbf{A}}]$	$\tilde{\mathbf{m}}$ in $\tilde{\mathbf{E}}$	$\mathbf{P}_{\mathrm{em}} = \mu_0\epsilon_0(\tilde{\mathbf{m}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{A}}$

3. Hidden momentum

Now, there is a general theorem [3,4] in special relativity that says 'if the centre of energy of a localized system is at rest, then the total momentum is zero.' In the cases, we are considering (stationary dipoles in static fields) the centre of energy is certainly not moving, and yet the *field* momentum is *not* zero, as we have seen. Evidently, there must be some *other* momentum, equal and opposite to $\mathbf{P}_{\mathrm{em}}$. This other momentum has come to be called 'hidden' momentum [5,6],[3] though there is nothing secret about it—in the present context, it is perfectly ordinary mechanical momentum, relativistic in nature, and 'hidden' only in the sense that it is not associated with motion of the object (here, the dipole) as a whole, but rather with its internally moving parts.

The most illuminating example of hidden momentum goes back to Penfield & Haus in the mid-1960s [8]. Imagine a rectangular loop of wire, carrying a steady current I in the presence of a uniform electrostatic field $\mathbf{E}$ (figure 1). Picture the current as a resistanceless flow of free positive

[3]For a history, and comprehensive references, see [7].

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

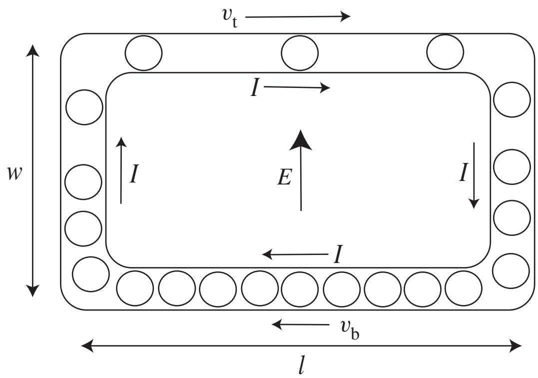

Figure 1. The Penfield–Haus model.

charges,[4] each with charge q and mass m. The electric field accelerates them as they ascend the left side, and slows them down as they descend the right side. Accordingly, their speed is greater along the top segment than at the bottom: $v_t > v_b$. On the other hand, they are further apart in the top segment, so there are more of them at the bottom: $N_b > N_t$. The current (which, remember, is constant around the loop) is

$$I = \frac{N_t q}{l} v_t = \frac{N_b q}{l} v_b \quad \Rightarrow \quad N_t v_t = N_b v_b = \frac{Il}{q}. \tag{3.1}$$

The net (relativistic) momentum of the charges—to the right—is

$$P_h = \gamma_t N_t m v_t - \gamma_b N_b m v_b = \frac{Ilm}{q}(\gamma_t - \gamma_b). \tag{3.2}$$

Now, the kinetic energy gained in ascending the left leg is equal to the work done by the electric force:

$$\gamma_t mc^2 - \gamma_b mc^2 = qEw \quad \Rightarrow \quad \gamma_t - \gamma_b = \frac{qEw}{mc^2}, \tag{3.3}$$

so

$$P_h = \left(\frac{Ilm}{q}\right)\left(\frac{qEw}{mc^2}\right) = \frac{1}{c^2}(Ilw)E. \tag{3.4}$$

But Ilw is the magnetic dipole moment of the loop; it points into the page. So

$$\mathbf{P}_h = \frac{1}{c^2}(\mathbf{m} \times \mathbf{E}) \quad [\mathbf{m} \text{ in } \mathbf{E}]. \tag{3.5}$$

This is the hidden momentum of the configuration—it is nothing but the net mechanical momentum of the charges constituting the current. It is independent of the *size* of the dipole (as long as $\mathbf{E}$ is constant over its area), so it applies in particular to an ideal (point) dipole. And it is just right to cancel the field momentum (equation (2.15))

$$\mathbf{P}_{em} = -\mu_0\epsilon_0(\mathbf{m} \times \mathbf{E}),$$

as required by the centre of energy theorem.

Notice that the Penfield–Haus mechanism applies to particles *in motion*. If, for example, this were an *anomalous* magnetic dipole, made from monopoles at rest, there would be *no* hidden momentum (in a standard electric field):

$$\mathbf{P}_h = \mathbf{0} \quad [\tilde{\mathbf{m}} \text{ in } \mathbf{E}]. \tag{3.6}$$

But, in that case, there is no *field* momentum either (equation (2.16)), and the total is again zero.

[4]This is not, of course, a realistic model of an actual current-carrying wire, but it makes the essential point with a minimum of extraneous baggage. For other models, see [9,10].

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

What if $\mathbf{E}$ is *not* uniform over the current loop? Consider a segment $d\mathbf{l}$; its momentum is

$$\mathrm{d}\mathbf{P} = \gamma(\lambda_m\,\mathrm{d}l)\mathbf{v} = \gamma\frac{\lambda_m}{\lambda_e}\lambda_e v\,\mathrm{d}\mathbf{l} = \gamma(\alpha I)\,\mathrm{d}\mathbf{l}, \tag{3.7}$$

where λ_m is the mass (of the moving charges) per unit length, λ_e is their charge per unit length and α is the mass-to-charge ratio). For the whole loop, then,

$$\mathbf{P}_\mathrm{h} = \alpha I \oint \gamma\,\mathrm{d}\mathbf{l}. \tag{3.8}$$

Picking as the reference point for potential some convenient spot $\mathcal{O}$ on the loop,

$$\gamma(\mathbf{r}) = \gamma_0 + \int_{\mathcal{O}}^{\mathbf{r}} \frac{\mathrm{d}W}{mc^2} = \gamma_0 + \frac{q}{mc^2}\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot\mathrm{d}\mathbf{l} = \gamma_0 - \frac{1}{\alpha c^2}V(\mathbf{r}), \tag{3.9}$$

where γ_0 is the value at $\mathcal{O}$, and $\mathrm{d}W$ is the work done on a charge as it advances by $\mathrm{d}\mathbf{l}$ along the loop. But

$$\oint \gamma_0\,\mathrm{d}\mathbf{l} = \gamma_0 \oint \mathrm{d}\mathbf{l} = \mathbf{0}, \tag{3.10}$$

and hence[5]

$$\mathbf{P}_\mathrm{h} = -\frac{I}{c^2}\oint V(\mathbf{r})\,\mathrm{d}\mathbf{l}. \tag{3.11}$$

For example, if the source of the electric field is an ordinary electric dipole, $\mathbf{p}$, (I'm changing the reference point, but the closed line integral is independent of any added constant)

$$\mathbf{P}_\mathrm{h} = -\frac{I}{c^2}\frac{1}{4\pi\epsilon_0}\oint \frac{[\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}\mathbf{l} = -\frac{\mu_0}{4\pi}\oint \mathbf{I}\frac{[\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}l, \tag{3.12}$$

where $(\mathbf{r}-\mathbf{r}')$ is the vector from the dipole (at $\mathbf{r}'$) to the point $\mathbf{r}$. (It does not matter whether you associate the vector with I or with $\mathrm{d}l$, since they are in the same direction: $I\,\mathrm{d}\mathbf{l} = \mathbf{I}\,\mathrm{d}l$.) We can express this result in terms of the vector potential (due to the current loop) at the location of the electric dipole:

$$\mathbf{A}(\mathbf{r}') = \frac{\mu_0}{4\pi}\oint \frac{\mathbf{I}(\mathbf{r})}{|\mathbf{r}'-\mathbf{r}|}\,\mathrm{d}l. \tag{3.13}$$

Thus

$$[(\mathbf{p}\cdot\boldsymbol{\nabla}')\mathbf{A}(\mathbf{r}')]_i = \frac{\mu_0}{4\pi}\oint I_i p_j \nabla'_j\left(\frac{1}{|\mathbf{r}'-\mathbf{r}|}\right)\mathrm{d}l = -\frac{\mu_0}{4\pi}\oint I_i p_j \frac{(\mathbf{r}'-\mathbf{r})_j}{|\mathbf{r}'-\mathbf{r}|^3}\,\mathrm{d}l, \tag{3.14}$$

or

$$[(\mathbf{p}\cdot\boldsymbol{\nabla}')\mathbf{A}(\mathbf{r}')] = -\frac{\mu_0}{4\pi}\oint \mathbf{I}\frac{[\mathbf{p}\cdot(\mathbf{r}'-\mathbf{r})]}{|\mathbf{r}'-\mathbf{r}|^3}\,\mathrm{d}l. \tag{3.15}$$

So the hidden momentum is

$$\mathbf{P}_\mathrm{h} = -(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{A}, \quad [\mathbf{p}\text{ in }\mathbf{B}], \tag{3.16}$$

which once again is just right to cancel the field momentum (equation (2.12)).

The same argument applies to an anomalous *magnetic* dipole in an anomalous *electric* field, except that what does the work is now the magnetic force ($\mathbf{F} = \tilde{q}\tilde{\mathbf{B}}$) acting

[5]For volume currents equation (3.11) becomes $\mathbf{P}_\mathrm{h} = -(I/c^2)\int V\mathbf{J}\,\mathrm{d}\tau$, and we see immediately that it cancels the field momentum (equation (2.2)). If $\mathbf{E}$ is *uniform* over the current region, then $V(\mathbf{r}) = V(\mathbf{0}) - \mathbf{E}\cdot\mathbf{r}$, and (using equation (1.20)) $\mathbf{P}_{\mathrm{h}_j} = \mu_0\epsilon_0 E_i \int r_i J_j\,\mathrm{d}\tau = \mu_0\epsilon_0 E_i \epsilon_{ijk} m_k = \mu_0\epsilon_0(\mathbf{m}\times\mathbf{E})_j$, so we recover equation (3.5).

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

on the particles in the monopole current loop:

$$\mathbf{P}_\mathrm{h} = -\frac{1}{c^2}(\tilde{\mathbf{m}} \cdot \boldsymbol{\nabla})\tilde{\mathbf{A}} \quad [\tilde{\mathbf{m}} \text{ in } \tilde{\mathbf{E}}]. \tag{3.17}$$

Alternatively, you can get equation (3.17) by applying the duality transformation to equation (3.16). Likewise, from equations (3.5) and (3.6),

$$\mathbf{P}_\mathrm{h} = -(\tilde{\mathbf{p}} \times \tilde{\mathbf{B}}) \quad [\tilde{\mathbf{p}} \text{ in } \tilde{\mathbf{B}}], \tag{3.18}$$

and

$$\mathbf{P}_\mathrm{h} = \mathbf{0} \quad [\mathbf{p} \text{ in } \tilde{\mathbf{B}}]. \tag{3.19}$$

The original Penfield–Haus model made no reference to the *source* of the electrostatic field—they presumably took it to be some collection of stationary electric charges (perhaps in the form of a surrounding parallel-plate capacitor). The hidden momentum in the magnetic dipole itself would be the same (equation (3.5)) if the electric field were due to a current of magnetic monopoles ($(1/c^2)(\mathbf{m} \times \tilde{\mathbf{E}})$). However, in that case there would *also* be hidden momentum residing in the monopole current (the monopoles accelerating and decelerating in response to the magnetic field of the electric current loop). The latter is given by equation (3.17) (for the momentum *in the monopole current loop* it does not matter whether the source of the magnetic field is standard or anomalous). Combining the two we get

$$\mathbf{P}_\mathrm{h} = \frac{1}{c^2}[(\mathbf{m} \times \tilde{\mathbf{E}}) - (\mathbf{m} \cdot \boldsymbol{\nabla})\tilde{\mathbf{A}}] \quad (\mathbf{m} \text{ in } \tilde{\mathbf{E}}). \tag{3.20}$$

By the same argument (or by invoking the duality transformation)

$$\mathbf{P}_\mathrm{h} = -[(\tilde{\mathbf{p}} \times \mathbf{B}) + (\tilde{\mathbf{p}} \cdot \boldsymbol{\nabla})\mathbf{A}] \quad (\tilde{\mathbf{p}} \text{ in } \mathbf{B}). \tag{3.21}$$

The following catalogue summarizes these results:

$\mathbf{p}$ in $\mathbf{B}$	$\mathbf{P}_\mathrm{h} = -(\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{A}$	resides in source of $\mathbf{B}$
$\mathbf{m}$ in $\mathbf{E}$	$\mathbf{P}_\mathrm{h} = \frac{1}{c^2}(\mathbf{m} \times \mathbf{E})$	resides in $\mathbf{m}$
$\mathbf{p}$ in $\tilde{\mathbf{B}}$	$\mathbf{P}_\mathrm{h} = \mathbf{0}$	(nothing moving)
$\mathbf{m}$ in $\tilde{\mathbf{E}}$	$\mathbf{P}_\mathrm{h} = \frac{1}{c^2}[\mathbf{m} \times \tilde{\mathbf{E}} - (\mathbf{m} \cdot \boldsymbol{\nabla})\tilde{\mathbf{A}}]$	resides in $\mathbf{m}$ and source of $\tilde{\mathbf{E}}$
$\tilde{\mathbf{p}}$ in $\mathbf{B}$	$\mathbf{P}_\mathrm{h} = -[(\tilde{\mathbf{p}} \times \mathbf{B}) + (\tilde{\mathbf{p}} \cdot \boldsymbol{\nabla})\mathbf{A}]$	resides in $\tilde{\mathbf{p}}$ and source of $\mathbf{B}$
$\tilde{\mathbf{m}}$ in $\mathbf{E}$	$\mathbf{P}_\mathrm{h} = \mathbf{0}$	(nothing moving)
$\tilde{\mathbf{p}}$ in $\tilde{\mathbf{B}}$	$\mathbf{P}_\mathrm{h} = -(\tilde{\mathbf{p}} \times \tilde{\mathbf{B}})$	resides in $\tilde{\mathbf{p}}$
$\tilde{\mathbf{m}}$ in $\tilde{\mathbf{E}}$	$\mathbf{P}_\mathrm{h} = -\frac{1}{c^2}(\tilde{\mathbf{m}} \cdot \boldsymbol{\nabla})\tilde{\mathbf{A}}$	resides in source of $\mathbf{E}$

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

In each case, the hidden momentum is just right to cancel the field momentum.

4. Interacting dipoles

As an application, suppose that the field is itself due to another dipole. There are four possibilities: (1) $\mathbf{p}$ and $\mathbf{m}$, (2) $\mathbf{p}$ and $\tilde{\mathbf{m}}$, (3) $\tilde{\mathbf{p}}$ and $\mathbf{m}$ and (4) $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{m}}$. We could regard the first as a standard magnetic dipole in the electric field of a standard electric dipole:

$$\begin{aligned}\mathbf{P}_{\mathrm{h}} &= \frac{1}{c^2}(\mathbf{m}\times\mathbf{E}) = \mu_0\epsilon_0\mathbf{m}\times\left\{\frac{1}{4\pi\epsilon_0}\frac{[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}]}{r^3} - \frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r})\right\}\\ &= \frac{\mu_0}{4\pi r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})(\mathbf{m}\times\hat{\mathbf{r}}) - (\mathbf{m}\times\mathbf{p})] - \frac{\mu_0}{3}(\mathbf{m}\times\mathbf{p})\,\delta^3(\mathbf{r}),\end{aligned} \tag{4.1}$$

(where $\mathbf{r}$ is the vector from one dipole to the other), or as a standard electric dipole in the magnetic field of a standard magnetic dipole:

$$\left.\begin{aligned} &\mathbf{P}_{\mathrm{h}} = -(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{A} = -(\mathbf{p}\cdot\boldsymbol{\nabla})\frac{\mu_0}{4\pi}\frac{(\mathbf{m}\times\mathbf{r})}{r^3},\\ \text{or}\quad &P_{\mathrm{h}i} = -\frac{\mu_0}{4\pi}\epsilon_{ijk}p_l m_j\nabla_l\left(\frac{r_k}{r^3}\right) = -\frac{\mu_0}{4\pi}\epsilon_{ijk}p_l m_j\left[\frac{1}{r^3}\left(\delta_{lk}-3\frac{r_l r_k}{r^2}\right)+\frac{4\pi}{3}\delta_{lk}\,\delta^3(\mathbf{r})\right]\\ &\qquad = -\frac{\mu_0}{4\pi}\left\{\frac{[-3(\mathbf{p}\cdot\hat{\mathbf{r}})(\mathbf{m}\times\hat{\mathbf{r}})_i + (\mathbf{m}\times\mathbf{p})_i]}{r^3} + \frac{4\pi}{3}(\mathbf{m}\times\mathbf{p})_i\delta^3(\mathbf{r})\right\},\end{aligned}\right\} \tag{4.2}$$

and we recover equation (4.1).

In the same way, we obtain the hidden momentum in the other three cases. These results are summarized below:

$\mathbf{p},\mathbf{m}$	$\mathbf{P}_{\mathrm{h}} = \frac{\mu_0}{4\pi r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})(\mathbf{m}\times\hat{\mathbf{r}}) - (\mathbf{m}\times\mathbf{p})] + \frac{\mu_0}{3}(\mathbf{p}\times\mathbf{m})\,\delta^3(\mathbf{r})$	resides in $\mathbf{m}$
$\mathbf{p},\tilde{\mathbf{m}}$	$\mathbf{P}_{\mathrm{h}} = \mathbf{0}$	nothing moving
$\tilde{\mathbf{p}},\mathbf{m}$	$\mathbf{P}_{\mathrm{h}} = \frac{\mu_0}{4\pi r^3}\{3[(\tilde{\mathbf{p}}\times\mathbf{m})\cdot\hat{\mathbf{r}}]\hat{\mathbf{r}} - (\tilde{\mathbf{p}}\times\mathbf{m})\} - \frac{\mu_0}{3}(\tilde{\mathbf{p}}\times\mathbf{m})\,\delta^3(\mathbf{r})$	resides in $\tilde{\mathbf{p}}$ and $\mathbf{m}$
$\tilde{\mathbf{p}},\tilde{\mathbf{m}}$	$\mathbf{P}_{\mathrm{h}} = -\frac{\mu_0}{4\pi r^3}[3(\tilde{\mathbf{m}}\cdot\hat{\mathbf{r}})(\tilde{\mathbf{p}}\times\hat{\mathbf{r}}) - (\tilde{\mathbf{p}}\times\tilde{\mathbf{m}})] + \frac{\mu_0}{3}(\tilde{\mathbf{p}}\times\tilde{\mathbf{m}})\,\delta^3(\mathbf{r})$	resides in $\tilde{\mathbf{p}}$

5. Spherical shell models

In this paper, I have treated ideal (point) dipoles, whose fields include the subtle delta function terms. It is embarrassingly easy to get these 'contact' contributions wrong, and wise to check one's results using a finite model. What if we picture the dipoles as spherical shells, of radius R, carrying appropriate surface charges (σ) or currents ($\mathbf{K}$)?[6] Letting $v \equiv \frac{4}{3}\pi R^3$ be the volume of the sphere:

[6]If you prefer, think of them as uniformly polarized or uniformly magnetized solid spheres, but this raises diverting questions about the correct formula for the field momentum inside a material medium (Abraham versus Minkowski), which I would like to avoid.

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

1. *Standard electric dipole*, $\mathbf{p}$: $\sigma = (\mathbf{p}\cdot\hat{\mathbf{r}})/v$,

$$V(\mathbf{r}) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{(\mathbf{p}\cdot\hat{\mathbf{r}})}{r^2}, & (r > R), \\ \dfrac{(\mathbf{p}\cdot\mathbf{r})}{3\epsilon_0 v}, & (r < R) \end{cases} \tag{5.1}$$

and therefore

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]}{r^3}, & (r > R), \\ -\dfrac{\mathbf{p}}{3\epsilon_0 v}, & (r < R). \end{cases} \tag{5.2}$$

2. *Standard magnetic dipole*, $\mathbf{m}$: $\mathbf{K} = (\mathbf{m}\times\hat{\mathbf{r}})/v$,

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{(\mathbf{m}\times\hat{\mathbf{r}})}{r^2}, & (r > R), \\ \dfrac{\mu_0(\mathbf{m}\times\mathbf{r})}{3v}, & (r < R) \end{cases} \tag{5.3}$$

and therefore

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]}{r^3}, & (r > R), \\ \dfrac{2\mu_0\mathbf{m}}{3v}, & (r < R). \end{cases} \tag{5.4}$$

3. *Anomalous magnetic dipole*, $\tilde{\mathbf{m}}$: $\tilde{\sigma} = (\tilde{\mathbf{m}}\cdot\hat{\mathbf{r}})/v$,

$$\tilde{V}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{(\tilde{\mathbf{m}}\cdot\hat{\mathbf{r}})}{r^2}, & (r > R), \\ \dfrac{\mu_0(\tilde{\mathbf{m}}\cdot\mathbf{r})}{3v}, & (r < R) \end{cases} \tag{5.5}$$

and therefore

$$\tilde{\mathbf{B}}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{[3(\tilde{\mathbf{m}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \tilde{\mathbf{m}}]}{r^3}, & (r > R), \\ -\dfrac{\mu_0\tilde{\mathbf{m}}}{3v}, & (r < R). \end{cases} \tag{5.6}$$

4. *Anomalous electric dipole*, $\tilde{\mathbf{p}}$: $\tilde{\mathbf{K}} = -c^2[(\tilde{\mathbf{p}}\times\hat{\mathbf{r}})/v]$,

$$\tilde{\mathbf{A}}(\mathbf{r}) = \begin{cases} -\dfrac{1}{4\pi\epsilon_0}\dfrac{(\tilde{\mathbf{p}}\times\hat{\mathbf{r}})}{r^2}, & (r > R), \\ -\dfrac{(\tilde{\mathbf{p}}\times\mathbf{r})}{3\epsilon_0 v}, & (r < R) \end{cases} \tag{5.7}$$

and therefore

$$\tilde{\mathbf{E}}(\mathbf{r}) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{[3(\tilde{\mathbf{p}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \tilde{\mathbf{p}}]}{r^3}, & (r > R), \\ \dfrac{2\tilde{\mathbf{p}}}{3\epsilon_0 v}, & (r < R). \end{cases} \tag{5.8}$$

As $R \to 0$, $1/v \to \delta^3(\mathbf{r})$, and we recover the ideal dipole fields (equations (1.33)–(1.36)).

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

Now suppose the sphere is *both* an electric dipole (either kind) *and* a magnetic dipole (either type).[7] Let us first calculate the field momentum for each combination. The external contribution ($r > R$) is the same for all of them; letting $\mathbf{a} \equiv (\mathbf{p} \times \mathbf{m})$:

$$\mathbf{P}_{\text{em}}^{\text{out}} = \epsilon_0 \frac{1}{4\pi\epsilon_0}\frac{\mu_0}{4\pi}\int \frac{[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] \times [3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]}{r^6}\,\mathrm{d}\tau$$
$$= \frac{\mu_0}{(4\pi)^2}\int \frac{[3(\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - 2\mathbf{a}]}{r^6}\,\mathrm{d}\tau. \tag{5.9}$$

Setting the z axis along $\mathbf{a}$, so that $\mathbf{a} = a\hat{\mathbf{z}}$, $\mathbf{a}\cdot\hat{\mathbf{r}} = a\cos\theta$ and $\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$, the ϕ integral kills the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components, leaving

$$\mathbf{P}_{\text{em}}^{\text{out}} = \frac{\mu_0}{(4\pi)^2}(a\hat{\mathbf{z}})(2\pi)\int_R^\infty \frac{1}{r^4}\,\mathrm{d}r\int_0^\pi (3\cos^2\theta - 2)\sin\theta\,\mathrm{d}\theta \tag{5.10}$$
$$= -\frac{\mu_0}{4\pi}\frac{(\mathbf{p}\times\mathbf{m})}{3R^3}. \tag{5.11}$$

The internal contribution ($r < R$) is trivial (since the fields are uniform), but different for the (four) different configurations:

1. <u>$\mathbf{p}$ and $\mathbf{m}$</u>: $\mathbf{P}_{\text{em}}^{\text{in}} = \epsilon_0(-\mathbf{p}/3\epsilon_0 v) \times (2\mu_0\mathbf{m}/3v)v = -2\mu_0(\mathbf{p}\times\mathbf{m})/9v$.
$$\text{So } \mathbf{P}_{\text{em}} = -\frac{\mu_0}{4\pi}\frac{(\mathbf{p}\times\mathbf{m})}{R^3}. \tag{5.12}$$
2. <u>$\mathbf{p}$ and $\tilde{\mathbf{m}}$</u>: $\mathbf{P}_{\text{em}}^{\text{in}} = \epsilon_0(-\mathbf{p}/3\epsilon_0 v) \times (-\mu_0\tilde{\mathbf{m}}/3v)v = (\mu_0/9v)(\mathbf{p}\times\tilde{\mathbf{m}})$.
$$\text{So } \mathbf{P}_{\text{em}} = \mathbf{0}. \tag{5.13}$$
3. <u>$\tilde{\mathbf{p}}$ and $\mathbf{m}$</u>: $\mathbf{P}_{\text{em}}^{\text{in}} = \epsilon_0(2\tilde{\mathbf{p}}/3\epsilon_0 v) \times (2\mu_0\mathbf{m}/3v)v = 4\mu_0(\tilde{\mathbf{p}}\times\mathbf{m})/9v$.
$$\text{So } \mathbf{P}_{\text{em}} = \frac{\mu_0}{4\pi}\frac{(\tilde{\mathbf{p}}\times\mathbf{m})}{R^3}. \tag{5.14}$$
4. <u>$\tilde{\mathbf{p}}$ and $\tilde{\mathbf{m}}$</u>: $\mathbf{P}_{\text{em}}^{\text{in}} = \epsilon_0(2\tilde{\mathbf{p}}/3\epsilon_0 v) \times (-\mu_0\tilde{\mathbf{m}}/3v)v = -(2\mu_0/9v)(\tilde{\mathbf{p}}\times\tilde{\mathbf{m}})$.
$$\text{So } \mathbf{P}_{\text{em}} = -\frac{\mu_0}{4\pi}\frac{(\tilde{\mathbf{p}}\times\tilde{\mathbf{m}})}{R^3}. \tag{5.15}$$

Now let's calculate the hidden momentum in each configuration:

1. <u>$\mathbf{p}$ and $\mathbf{m}$</u>: I would *like* to use the Penfield–Haus formula (equation (3.5)), but that assumes the electric field is uniform over the current region. In this case, the field *is* uniform *inside* the sphere, but right at the surface (where the current is located) $\mathbf{E}$ is discontinuous, and the external field is *not* uniform. You can finesse this problem by a trick: make the magnetic sphere ever-so-slightly *smaller* than the electric sphere; then the electric field really *is* uniform over the current, and we get[8]
$$\mathbf{P}_{\text{h}} = \frac{1}{c^2}(\mathbf{m}\times\mathbf{E}) = \mu_0\epsilon_0\left[\mathbf{m}\times\left(\frac{-\mathbf{p}}{3\epsilon_0 v}\right)\right] = \frac{\mu_0}{4\pi}\frac{(\mathbf{p}\times\mathbf{m})}{R^3}. \tag{5.16}$$
2. <u>$\tilde{\mathbf{p}}$ and $\tilde{\mathbf{m}}$</u>: using the duality transformation and equation (5.19)
$$\mathbf{P}_{\text{h}} = \frac{\mu_0}{4\pi}\frac{(\tilde{\mathbf{p}}\times\tilde{\mathbf{m}})}{R^3}. \tag{5.17}$$
3. <u>$\mathbf{p}$ and $\tilde{\mathbf{m}}$</u>: nothing is moving, so
$$\mathbf{P}_{\text{h}} = \mathbf{0}. \tag{5.18}$$

[7] The 'standard' case ($\mathbf{p}$ and $\mathbf{m}$) was introduced by Romer [11].

[8] If this bothers you, go back to equation (3.11) (or rather, its analogue for surface currents), $\mathbf{P}_{\text{h}} = -(1/c^2)\int V\mathbf{K}\,\mathrm{d}a$; you get the same answer either way.

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

4. $\tilde{\mathbf{p}}$ and $\mathbf{m}$: In this case, there is a hidden momentum in *both* spheres. Using the trick (making the magnetic sphere slightly smaller than the electric sphere—you can do it the other way, of course, but you get the same answer), the hidden momentum in the electric current is

$$\mathbf{P}_{\mathrm{h}}^{e} = \frac{1}{c^2}(\mathbf{m} \times \tilde{\mathbf{E}}) = \mu_0\epsilon_0 \left[\mathbf{m} \times \left(\frac{2\tilde{\mathbf{p}}}{3\epsilon_0 v}\right)\right] = -2\,\frac{\mu_0}{4\pi}\frac{(\tilde{\mathbf{p}} \times \mathbf{m})}{R^3}. \tag{5.19}$$

But the magnetic field in the vicinity of the monopole current is *not* uniform, and we must use equation (3.8) (or rather, its analogue for a monopole current ($\tilde{I}$) in a standard magnetic field $\mathbf{B}$):

$$\mathbf{P} = \tilde{\alpha}\tilde{I}\oint \gamma\, \mathrm{d}\mathbf{l}. \tag{5.20}$$

In this case, equation (3.9) becomes

$$\gamma(\mathbf{r}) = \gamma_0 + \frac{1}{\tilde{\alpha}c^2}\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{B}\cdot \mathrm{d}\mathbf{l}. \tag{5.21}$$

We might as well choose our axes so that $\tilde{\mathbf{p}}$ points in the z direction. I will first calculate the hidden momentum in a single ring of monopole current, $\tilde{I}$, at $z = R\cos\theta$, with radius $R\sin\theta$.

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]}{R^3}, \quad \mathrm{d}\mathbf{l} = R\sin\theta\, \mathrm{d}\phi\hat{\boldsymbol{\phi}}, \tag{5.22}$$

and (setting the reference point directly above the x axis),

$$\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{B}\cdot \mathrm{d}\mathbf{l} = \frac{\mu_0}{4\pi}\frac{R\sin\theta}{R^3}\int_0^{\phi}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]\cdot\hat{\boldsymbol{\phi}}\,\mathrm{d}\phi = -\frac{\mu_0\sin\theta}{4\pi R^2}\int_0^{\phi}(\mathbf{m}\cdot\hat{\boldsymbol{\phi}})\,\mathrm{d}\phi.$$

Now

$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}, \quad \text{so } \mathbf{m}\cdot\hat{\boldsymbol{\phi}} = -m_x\sin\phi + m_y\cos\phi,$$

and therefore

$$\begin{aligned}\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{B}\cdot \mathrm{d}\mathbf{l} &= -\frac{\mu_0\sin\theta}{4\pi R^2}\int_0^{\phi}(-m_x\sin\phi + m_y\cos\phi)\,\mathrm{d}\phi \\ &= -\frac{\mu_0\sin\theta}{4\pi R^2}\left[m_x(\cos\phi - 1) + m_y\sin\phi\right].\end{aligned} \tag{5.23}$$

The hidden momentum in this ring is

$$\begin{aligned}\mathbf{P}_{\mathrm{h}}^{\mathrm{ring}} &= \frac{\tilde{I}}{c^2}\left(-\frac{\mu_0\sin\theta}{4\pi R^2}\right)\oint\left[m_x(\cos\phi - 1) + m_y\sin\phi\right]R\sin\theta\,\mathrm{d}\phi\hat{\boldsymbol{\phi}} \\ &= -\frac{\tilde{I}\mu_0\sin^2\theta}{4\pi c^2 R}\int_0^{2\pi}\left[m_x(\cos\phi - 1) + m_y\sin\phi\right](-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}})\,\mathrm{d}\phi \\ &= -\frac{\tilde{I}\mu_0\sin^2\theta}{4\pi c^2 R}\left(-\pi m_y\hat{\mathbf{x}} + \pi m_x\hat{\mathbf{y}}\right).\end{aligned} \tag{5.24}$$

Now we integrate over all the rings that cover the monopole current sphere, using $\tilde{I} \to -|\tilde{\mathbf{K}}|R\,\mathrm{d}\theta$ and $\tilde{\mathbf{K}} = -c^2(\tilde{\mathbf{p}} \times \hat{\mathbf{r}})/v$

$$\begin{aligned}\mathbf{P}_{\mathrm{h}}^{\mathrm{m}} &= \frac{\mu_0}{4c^2}(-m_y\hat{\mathbf{x}} + m_x\hat{\mathbf{y}})\int_0^{\pi}\sin^2\theta\,\frac{c^2|\tilde{\mathbf{p}}\times\hat{\mathbf{r}}|}{v}\,\mathrm{d}\theta \\ &= \frac{\mu_0}{4v}\left(-m_y\hat{\mathbf{x}} + m_x\hat{\mathbf{y}}\right)\tilde{p}\int_0^{\pi}\sin^3\theta\,\mathrm{d}\theta = \frac{\mu_0}{4v}\left(-m_y\hat{\mathbf{x}} + m_x\hat{\mathbf{y}}\right)\tilde{p}\left(\frac{4}{3}\right) \\ &= \frac{\mu_0}{3v}(\tilde{\mathbf{p}}\times\mathbf{m}) = \frac{\mu_0}{4\pi}\frac{(\tilde{\mathbf{p}}\times\mathbf{m})}{R^3}.\end{aligned} \tag{5.25}$$

Finally, combining equations (5.19) and (5.25),

$$\mathbf{P}_{\mathrm{h}} = \mathbf{P}_{\mathrm{h}}^{e} + \mathbf{P}_{\mathrm{h}}^{m} = -\frac{\mu_0}{4\pi}\frac{(\tilde{\mathbf{p}}\times\mathbf{m})}{R^3}. \tag{5.26}$$

Downloaded from http://rsta.royalsocietypublishing.org/ on October 31, 2018

These results confirm the contact terms in the table in §4.[9] As always, the hidden momentum is equal and opposite to the field momentum. There is nothing *surprising* in any of this, but it is gratifying to see it work out in explicit detail.

Data accessibility. This article has no additional data.

Competing interests. I declare I have no competing interests.

Funding. No funding has been received for this article.

Acknowledgements. I thank Vladimir Hnizdo, Kirk McDonald and Pablo Saldanha for illuminating correspondence.

References

1. Griffiths DJ. 1992 Dipoles at rest. *Am. J. Phys.* **60**, 979–987. (doi:10.1119/1.17001)
2. Frahm CP. 1983 Some novel delta-function identities. *Am. J. Phys.* **51**, 826–829. (doi:10.1119/1.13127)
3. Coleman S, Van Vleck JH. 1968 Origin of 'hidden momentum forces' on magnets. *Phys. Rev.* **171**, 1370–1375. (doi:10.1103/PhysRev.171.1370)
4. Calkin MG. 1971 Linear momentum of the source of a static electromagnetic field. *Am. J. Phys.* **39**, 513–516. (doi:10.1119/1.1986204)
5. Shockley W, James RP. 1967 'Try simplest cases' discovery of 'hidden momentum' forces on 'magnetic currents'. *Phys. Rev. Lett.* **18**, 876–879. (doi:10.1103/PhysRevLett.18.876)
6. Costa de Beauregard O. 1967 A new law in electrodynamics. *Phys. Lett.* **A24**, 177–178. (doi:10.1016/0375-9601(67)90752-9)
7. McDonald KT. 2018 See http://physics.princeton.edu/~mcdonald/examples/hiddendef.pdf.
8. Penfield Jr P, Haus HA. 1967 *Electrodynamics of moving media*, pp. 214–216. Cambridge, MA: M.I.T. Press.
9. Vaidman L. 1990 Torque and force on a magnetic dipole. *Am. J. Phys.* **58**, 978–983. (doi:10.1119/1.16260)
10. Hnizdo V. 1997 Hidden momentum of a relativistic fluid carrying current in an external electric field. *Am. J. Phys.* **65**, 92–94. (doi:10.1119/1.18500)
11. Romer RH. 1995 Question #26. Electromagnetic field momentum. *Am. J. Phys.* **63**, 777–779. (doi:10.1119/1.18075)

[9]Since the two spheres coincide we are *only* checking the contact term. On the other hand, if we *separate* the spheres (by a distance greater than the sum of their radii) there is really nothing to check, since the fields are precisely those of an ideal dipole.